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Note on the long-wave limit of the virtual-mass coefficient for a half-immersed circular cylinder heaving on water of finite depth

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This note provides numerical values for the long-wave limit of the virtual-mass coefficient relating to the heaving motion of a half-immersed circular cylinder on water of finite depth, found analytically by Ursell in the preceding paper; some preliminary analysis is needed, however.

The problem where time-harmonic gravity waves are generated in water of finite constant depth by small vertical oscillations of a half-immersed circular cylinder has received considerable attention. Most recently, Ursell (1976) considered the long-wave asymptotic motion, his basic intention being to determine whether or not the virtual-mass coefficient is finite in the long-wave limit. After some complicated mathematical analysis, it was found that this coefficient is in fact finite, and an analytical form was obtained; this depends on a certain limit potential which was given in infinite-series form, but for which the coefficients were undetermined. The purpose of this note is to show that these coefficients may be found as the solution of an infinite system of linear equations and may be computed for any geometrical situation, so that then numerical values for the long-wave limit of the virtual-mass coefficient may be computed from an infinite-series form depending on the coefficients thus found. These values are expected to be helpful to other workers interested in long-wave asymptotic calculations. The idea used is similar to that in Rhodes-Robinson (1970), where numerical values were included for short-wave asymptotic motion, and details now follow.

The long-wave asymptotic value of the virtual-mass coefficient for a half-immersed circular cylinder of radius a heaving on water of finite constant depth h with angular frequency $\sigma = (gK)^{\frac{1}{2}}$ was determined by Ursell (1976) in the integral form

$$\text{virtual-mass coefficient} \sim -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} [\hat{B}(a\sin\theta, a\cos\theta; H) - H]\cos\theta\, d\theta + O(Kh)$$

as $Kh \to 0$, where we let $H = a/h$ so that $0 < H < 1$; using rectangular co-ordinates (x, y) and related polar co-ordinates (r, θ), the symmetric potential $\hat{B}(x, y)$ whose value on the cylinder $r = a$, $|\theta| < \frac{1}{2}\pi$ is involved in this expression uniquely satisfies (for $0 < H < 1$) Laplace's equation in the fluid region subject to the boundary conditions $\hat{B}_y = 0$ on the free surface $y = 0$, $|x| > a$ and bottom $y = h$,

$a\hat{B}_r = \cos\theta$ on the cylinder $r = a$, $|\theta| < \frac{1}{2}\pi$, and $\hat{B} - |x|/h \to 0$ as $|x| \to \infty$. Further, this has an expansion (for $r < 2h$ at least) of the form

$$\hat{B}(x,y) = \frac{2}{\pi}\hat{F}(x,y) + \sum_{n=1}^{\infty} a^{2n}\hat{B}_{2n}\hat{F}_{2n}(x,y)$$

in terms of the basic potentials

$$\hat{F} = \tfrac{1}{2}\log\left[2(\cosh \pi x/h - \cos \pi y/h)\right]$$

$$= \log \pi r/h - \sum_{s=1}^{\infty}\frac{1}{s}(r/2h)^{2s}\zeta(2s)\cos 2s\theta$$

and

$$\hat{F}_{2n} = \frac{\cos 2n\theta}{r^{2n}} + \frac{2}{(2n-1)!}(1/2h)^{2n}\sum_{s=0}^{\infty}\frac{(2n+2s-1)!}{(2s)!}(r/2h)^{2s}\zeta(2n+2s)\cos 2s\theta$$

($n = 1, 2, \ldots$), which are respectively a source potential and an infinite set of multipole potentials which are harmonic on the strip $0 < y < h$, $|x| < \infty$ and satisfy the boundary condition for zero normal velocity on $y = 0, h$; also $\hat{F} - \log r$ and $\hat{F}_{2n} - \cos 2n\theta/r^{2n}$ are bounded as $r \to 0$, and $\hat{F} - \pi|x|/2h$, $\hat{F}_{2n} \to 0$ as $|x| \to \infty$ (these expansions involve the Riemann zeta function for integral arguments).

The coefficients $\hat{B}_{2n}(H)$ are dimensionless and are determined *in principle* through application of the boundary condition $a\hat{B}_r(a\sin\theta, a\cos\theta) = \cos\theta$. Moreover, if computations of the long-wave limit of the virtual-mass coefficient are to be made, a procedure must be formulated *precisely* for determining at least numerical values explicitly for $n = 1, 2, \ldots$, so that $\hat{B}$ may be regarded as completely determined for any value of H taken; we now give details of this, using the expansions above which are suitable.

On applying the aforementioned boundary condition, we have

$$\cos\theta = \frac{2}{\pi} - \frac{4}{\pi}\sum_{s=1}^{\infty}(\tfrac{1}{2}H)^{2s}\zeta(2s)\cos 2s\theta + 2\sum_{n=1}^{\infty}\hat{B}_{2n}\left[-n\cos 2n\theta + \frac{(\frac{1}{2}H)^{2n}}{(2n-1)!}\sum_{s=1}^{\infty}\frac{(2n+2s-1)!}{(2s-1)!}(\tfrac{1}{2}H)^{2s}\zeta(2n+2s)\cos 2s\theta\right]$$

$$= \frac{2}{\pi} - 2\sum_{s=1}^{\infty}\cos 2s\theta\left[\frac{2}{\pi}(\tfrac{1}{2}H)^{2s}\zeta(2s) + s\hat{B}_{2s} - \frac{(\frac{1}{2}H)^{2s}}{(2s-1)!}\sum_{n=1}^{\infty}\hat{B}_{2n}\frac{(2s+2n-1)!}{(2n-1)!}(\tfrac{1}{2}H)^{2n}\zeta(2s+2n)\right],$$

on rearranging. Now, this expression must coincide with the Fourier cosine series representation

$$\cos\theta = \frac{2}{\pi} + \frac{4}{\pi}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{4s^2-1}\cos 2s\theta$$

for the interval $|\theta| \leqslant \frac{1}{2}\pi$, so that by comparison of the coefficients† we obtain the equations

$$\frac{2}{\pi}\frac{(-1)^{s-1}}{4s^2-1} = -\frac{2}{\pi}(\tfrac{1}{2}H)^{2s}\zeta(2s) - s\hat{B}_{2s} + \frac{(\frac{1}{2}H)^{2s}}{(2s-1)!}\sum_{n=1}^{\infty}\hat{B}_{2n}\frac{(2s+2n-1)!}{(2n-1)!}(\tfrac{1}{2}H)^{2n}\zeta(2s+2n)$$

† Note that the terms outside the summation are identically equal already owing to the exact evaluation of the constant $\hat{B}_0$ by Ursell (1976).

H	Long-wave limit	H	Long-wave limit	H	Long-wave limit
0·01	2·9099	0·34	0·5278	0·67	0·6331
0·02	2·3609	0·35	0·5208	0·68	0·6466
0·03	2·0452	0·36	0·5145	0·69	0·6607
0·04	1·8250	0·37	0·5090	0·70	0·6756
0·05	1·6573	0·38	0·5042	0·71	0·6913
0·06	1·5227	0·39	0·5002	0·72	0·7077
0·07	1·4111	0·40	0·4969	0·73	0·7250
0·08	1·3163	0·41	0·4943	0·74	0·7432
0·09	1·2343	0·42	0·4923	0·75	0·7622
0·10	1·1625	0·43	0·4909	0·76	0·7823
0·11	1·0989	0·44	0·4902	0·77	0·8033
0·12	1·0422	0·45	0·4901	0·78	0·8255
0·13	0·9912	0·46	0·4906	0·79	0·8488
0·14	0·9451	0·47	0·4917	0·80	0·8733
0·15	0·9032	0·48	0·4933	0·81	0·8992
0·16	0·8650	0·49	0·4956	0·82	0·9265
0·17	0·8302	0·50	0·4984	0·83	0·9554
0·18	0·7982	0·51	0·5017	0·84	0·9859
0·19	0·7688	0·52	0·5057	0·85	1·0184
0·20	0·7418	0·53	0·5102	0·86	1·0529
0·21	0·7170	0·54	0·5152	0·87	1·0897
0·22	0·6940	0·55	0·5208	0·88	1·1290
0·23	0·6729	0·56	0·5269	0·89	1·1713
0·24	0·6534	0·57	0·5337	0·90	1·2170
0·25	0·6354	0·58	0·5409	0·91	1·2666
0·26	0·6189	0·59	0·5488	0·92	1·3207
0·27	0·6036	0·60	0·5572	0·93	1·3805
0·28	0·5896	0·61	0·5662	0·94	1·4471
0·29	0·5768	0·62	0·5758	0·95	1·5225
0·30	0·5650	0·63	0·5860	0·96	1·6096
0·31	0·5543	0·64	0·5968	0·97	1·7134
0·32	0·5446	0·65	0·6083	0·98	1·8439
0·33	0·5358	0·66	0·6204	0·99	2·0267

TABLE 1. Values of the long-wave limit of the virtual-mass coefficient

($s = 1, 2, \ldots$), i.e. $\hat{B}_{2s}$ satisfies the infinite linear system

$$s\hat{B}_{2s}(H) - \frac{(\frac{1}{2}H)^{2s}}{(2s-1)!}\sum_{n=1}^{\infty}\hat{B}_{2n}(H)\,\frac{(2s+2n-1)!}{(2n-1)!}\,(\tfrac{1}{2}H)^{2n}\,\zeta(2s+2n)$$
$$= -\frac{2}{\pi}\left[\frac{(-1)^{s-1}}{4s^2-1}+(\tfrac{1}{2}H)^{2s}\,\zeta(2s)\right]$$

for $s = 1, 2, \ldots$. This may be solved numerically to any required degree of accuracy by truncation to a finite system for any given H ($0 < H < 1$) since $\hat{B}_{2s} \to 0$ as $s \to \infty$.

Finally, the virtual-mass coefficient has the long-wave limit

$$-\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}[\hat{B}(a\sin\theta, a\cos\theta)-H]\cos\theta\, d\theta = \frac{4}{\pi}\left[\frac{2}{\pi}c_0 - \sum_{n=1}^{\infty}\hat{B}_{2n}c_n\right]$$

H	0·10	0·25	0·50
$\hat{B}_2$	−0·2148	−0·2289	−0·2848
$\hat{B}_4$	0·0212	0.0211	0·0192
$\hat{B}_6$	−0·0061	−0·0061	−0·0062
$\hat{B}_8$	0·0025	0·0025	0·0025
$\hat{B}_{10}$	−0·0013	−0·0013	−0·0013
$\hat{B}_{12}$	0·0007	0·0007	0·0007
$\hat{B}_{14}$	−0·0005	−0·0005	−0·0005
$\hat{B}_{16}$	0·0003	0·0003	0·0003
$\hat{B}_{18}$	−0·0002	−0·0002	−0·0002
$\hat{B}_{20}$	0·0002	0·0002	0·0002
$\hat{B}_{22}$	−0·0001	−0·0001	−0·0001
$\hat{B}_{24}$	0·0001	0·0001	0·0001
$\hat{B}_{26}$	−0·0001	−0·0001	−0·0001
$\hat{B}_{28}$	0·0001	0·0001	0·0001

TABLE 2. Some values of the expansion coefficients (only those non-zero to four decimal places are shown)

in series form, where

$$c_0(H) \equiv -\log \pi H + \tfrac{1}{2}\pi H + \sum_{s=1}^{\infty} \frac{1}{s} \frac{(-1)^{s-1}}{4s^2-1} (\tfrac{1}{2}H)^{2s} \zeta(2s)$$

and

$$c_n(H) \equiv \frac{(-1)^{n-1}}{4n^2-1} + 2\frac{(\frac{1}{2}H)^{2n}}{(2n-1)!} \sum_{s=0}^{\infty} \frac{(-1)^{s-1}}{4s^2-1} \frac{(2n+2s-1)!}{(2s)!} (\tfrac{1}{2}H)^{2s} \zeta(2n+2s) \quad (n = 1, 2, \ldots);$$

therefore this may be evaluated numerically by truncation for $0 < H < 1$ also, using the previously obtained values for $\hat{B}_{2n}$ $(n = 1, 2, \ldots)$, after the sums for c_n $(n = 0, 1, \ldots)$ have been evaluated in a similar way. [Note that the long-wave limit $\sim -(8/\pi^2) \log H$ as $H \to 0$.]

The results of the computations for the long-wave limit are presented in table 1 for the values† $H = 0{\cdot}01, 0{\cdot}02, \ldots, 0{\cdot}99$; results of the computations for the coefficients employed are shown in table 2 but only for $H = 0{\cdot}10$, 0·25 and 0·50. Note the existence of a *minimum* value for the long-wave limit, found to be 0·4901 for $H = 0{\cdot}4468$. The computations are all correct to four decimal places and were done by Burroughs B6700 computer at Victoria University of Wellington.

The values we have obtained for the long-wave limit would make interpolation possible in any calculations of the virtual-mass coefficient in the range of smaller Kh; the only calculations at present known to the author are those of Porter (1967, private communication) for $H = 0{\cdot}10$, 0·25 and 0·50 (the incorrect ones of Yu & Ursell 1961 excepted), but these go no lower than $Kh = 2$ and feasible interpolation is therefore out of the question. It ought to be pointed out, however, that even with suitably low calculations accuracy of interpolation may still be wanting since only the first-order approximation to the virtual-mass coefficient,

† Results for other values are available.

i.e. the long-wave limit, is known as $Kh \to 0$; a second-order approximation would alleviate this uncertainty, but any analytical attempt to obtain it would probably be difficult.

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REFERENCES

RHODES-ROBINSON, P. F. 1970 On the short-wave asymptotic motion due to a cylinder heaving on water of finite depth. II. *Proc. Camb. Phil. Soc.* **67**, 443–468.

URSELL, F. 1976 On the virtual-mass and damping coefficients for long waves in water of finite depth. *J. Fluid Mech.* **76**, 17–28.

YU, Y. S. & URSELL, F. 1961 Surface waves generated by an oscillating circular cylinder on water of finite depth: theory and experiment. *J. Fluid Mech.* **11**, 529–550.